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Quotients of Interval Effect Algebras

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Nearly every orthostructure that has been proposed as a model for a logic of propositions affiliated with a physical system can be represented as an interval effect algebra; that is, as the partial algebra under addition of an interval from zero to an order unit in a partially ordered Abelian group. If the system is in a state that precludes certain elements of such an interval, an appropriate quotient interval algebra can be constructed by factoring out the order-convex subgroup generated by the precluded elements. In this paper we launch a study of the resulting quotient effect algebras.

1. INTRODUCTION

Effect algebras (Foulis and Bennett, 1994; Greechie and Foulis, 1995), or what is essentially the same thing, D-posets (Kôpka, 1992; Navara and Pták, 1993; Kôpka and Chovanec, 1994), arose partly as an answer to the problem of representing fuzzy or unsharp events (Dalla Chiara and Giuntini, 1989; Giuntini and Greuling, 1989; Mesiar, 1993), partly in connection with positive-operator-valued (POV) measures in stochastic quantum mechanics (Schroeck and Foulis, 1990), and partly in response to the problem of forming tensor products of quantum logics (Randall and Foulis, 1981; Pulmannová, 1985; Kläy *et al.*, 1987; Bennett and Foulis, 1993; Dvurečenskij, 1994; Dvurečenskij and Pulmannová, 1994; Foulis *et al.*, 1994). Boolean algebras, orthomodular lattices, orthomodular posets, and orthoalgebras are special kinds of effect algebras. Effect algebras can be combined by forming Cartesian products, horizontal sums, and tensor products (Foulis *et al.*, 1994). However, until now, no comprehensive theory of quotients has been worked out,

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although a possible basis for such a theory has been sketched in Foulis *et al.* (1996).

Propositions about a physical system $\mathcal{S}$ tend to band together to form an effect algebra E, and the physical states of $\mathcal{S}$ give rise to probability measures on E. If $\mathcal{S}$ is in a physical state ψ with corresponding probability measure $\omega: E \to [0, 1] \subseteq \mathbb{R}$, then the propositions in $I = \omega^{-1}(0)$ are (probabilistically) impossible. In accordance with standard practices in classical mathematical logic, one ought to be able to form a suitable "quotient effect algebra" E/I by somehow "factoring out" the impossible propositions, thus obtaining a representation for the propositions affiliated with $\mathcal{S}$ when it is known to be in the state ψ.

Effect algebras that can be represented as the interval from zero to a positive element in a partially ordered Abelian group are called *interval effect algebras* (Bennett and Foulis, n.d.). These form a large and important subclass of the class of all effect algebras and, for this subclass, a natural notion of quotient suggests itself. In this paper, we launch a study of such quotients, showing that in some ways they are well behaved and in other ways they are not. It is hoped that our study will contribute toward a solution of the problem of formulating an appropriate general theory of quotients for effect algebras.

2. BASIC DEFINITIONS

The general notion of an effect algebra is as follows.

Definition 2.1. An *effect algebra* is a system $(E, \oplus, 0, u)$ consisting of a set E, a partially defined binary operation $\oplus$ on E, and two special elements $0, u \in E$ called the *zero* and the *unit* such that, for all $p, q, r \in E$:

(i) (*Associative Law*). If $p \oplus q$ and $(p \oplus q) \oplus r$ are defined, then $(q \oplus r)$ and $p \oplus (q \oplus r)$ are defined and

$$(p \oplus q) \oplus r = p \oplus (q \oplus r)$$

(ii) (*Commutative Law*). If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(iii) (*Orthosupplementation Law*). For each $p \in E$, there is a unique $q \in E$, such that $p \oplus q$ is defined and $p \oplus q = u$.

(iv) (*Zero-Unit Law*). If $u \oplus p$ is defined, then $p = 0$.

Unless confusion threatens, we say that E is an effect algebra when we really mean that $(E, \oplus, 0, u)$ is an effect algebra. Also, when we write an equation such as $p \oplus r = q$, we are asserting both that $p \oplus r$ is defined and that $p \oplus r = q$.

Example 2.2. Let $\mathcal{H}$ be a Hilbert space and let $\mathrm{E}(\mathcal{H})$ be the set of all self-adjoint operators A on $\mathcal{H}$ such that $0 \le A \le 1$. If $A, B \in \mathrm{E}(\mathcal{H})$, define $A \oplus B$ iff $A + B \le 1$, in which case $A \oplus B := A + B$. (We use $:=$ to mean equals by definition.) Then $\mathrm{E}(\mathcal{H})$ is an effect algebra called the *standard effect algebra* for $\mathcal{H}$.

For the remainder of this section, *we assume that E is an effect algebra with unit u*. If $p, q \in E$, we say that p is *orthogonal* to q and write $p \perp q$ iff $p \oplus q$ is defined. An element $p \in E$ is called *isotropic* iff $p \perp p$, that is, iff $p \oplus p$ is defined. The unique element $q \in E$ such that $p \oplus q = u$ is called the *orthosupplement* of p and written as $p' := q$. In Example 2.2, if $A \in \mathrm{E}(\mathcal{H})$, then $A' = 1 - A$ and A is isotropic iff $2A \le 1$.

The relation $\le$ defined on E by $p \le q$ iff $p \oplus r = q$ for some $r \in E$ is a partial order on E such that $0 \le p \le u$ for all $p \in E$. The mapping $'$: $E \to E$ is an order-reversing involution on E and $p \perp q \Leftrightarrow p \le q'$. We use standard order-theoretic terminology in connection with the poset $(E, \le)$. For instance, if 0 is the only element $r \in E$ for which $r \le p, q$, we say that p and q are *disjoint*.

An *orthoalgebra* (OA) is the same thing as an effect algebra with no nonzero isotropic elements (Foulis *et al.*, 1992), an *orthomodular poset* (OMP) is the same thing as an OA in which the $\oplus$-sum of two orthogonal elements is their supremum, and an *orthomodular lattice* (OML) is an OMP in which any two elements have a supremum (Foulis and Bennett, 1994). A *Boolean algebra* is the same thing as an OML in which disjoint elements are orthogonal.

A subset A of E is called a *sub-effect algebra* of E iff $0 \in A$, A is closed under orthosupplementation, and A is closed under the partial binary operation $\oplus$. Obviously, a sub-effect algebra A of an effect algebra E is an effect algebra in its own right under the restriction to A of the partial operation $\oplus$.

If F is a second effect algebra, then a mapping ϕ: $E \to F$ is called a *morphism* iff it preserves $\oplus$ and maps unity to unity. By definition, a *homomorphism* is a morphism that preserves disjoint pairs. A morphism ϕ: $E \to F$ is called a *monomorphism* iff for $p, q \in E$, $\phi(p) \le \phi(q) \Rightarrow p \le q$. An *isomorphism* is a surjective monomorphism.

If ϕ: $E \to F$ is a morphism, it is clear that $\phi(0) = 0$ and that, for $p \in E$, $\phi(p') = \phi(p)'$. Furthermore, $p \le q \Rightarrow \phi(p) \le \phi(q)$. Simple examples show that, in general, $\phi(E)$ need not be a subalgebra of F. If ϕ: $E \to F$ is a monomorphism, then $p \perp q$ iff $\phi(p) \perp \phi(q)$. Furthermore, if ϕ is a monomorphism, then $\phi(E)$ is a sub-effect algebra of F.

Definition 2.3. An *ideal* is a nonempty subset I of E such that for all $p, q \in E$ with $p \perp q$, $p \oplus q \in I \Leftrightarrow p, q \in I$.

Thus, a nonempty subset I of E is an ideal iff it is an order ideal in (E, $\leq$) and it is closed under the partial binary operation $\oplus$.

Example 2.4. If ϕ: $E \to F$ is a morphism, then $I := \phi^{-1}(0)$ is an ideal in E called the *effect kernel* of ϕ.

3. PARTIALLY ORDERED ABELIAN GROUPS

Although much of the material in this section is well known in the theory of ordered algebraic structures, we sketch it here for convenience and to establish our notation. Omitted proofs can be found, for instance, in Chapter 1 of Goodearl (1986). In what follows, *we assume that G is an additively written, partially ordered Abelian group* with *positive cone* $G^+ := \{g \in G \mid 0 \leq g\}$.

If H is a subgroup of G, then H is a partially ordered Abelian group in its own right under the restriction to H of the partial order $\leq$ on G. We refer to this as the *induced* partial order on H and to the corresponding positive cone $H^+ = H \cap G^+$ as the *induced positive cone*. For instance, in the additive group R of real numbers the *standard positive cone* R^+ consists of the real numbers that are nonnegative in the usual sense, the integers Z form an additive subgroup of R, and the induced positive cone is $Z^+ = Z \cap R^+$.

If $X \subseteq G$, we define $\langle X \rangle$ to be the subgroup of G generated by X and denote by $\langle a \rangle$ the cyclic subgroup of G generated by $a \in G$. A subgroup H of G is said to be *directed* iff for all a, $b \in H$, there exists $c \in H$ such that $a, b \leq c$.

Lemma 3.1. Let H be a subgroup of G and let $H^+ = H \cap G^+$. Then the following conditions are mutually equivalent:

(i) H is directed.

(ii) $H = H^+ - H^+$.

(iii) $H = \langle H^+ \rangle$.

(iv) $\exists X \subseteq G^+$ with $H = \langle X \rangle$.

Corollary 3.2. $G = \langle G^+ \rangle$ iff G is directed.

A subgroup H of G is said to be *order convex* iff, for all a, $b \in G$, $0 \leq a \leq b \in H \Rightarrow a \in H$.

Theorem 3.3. Let H be a subgroup of G. Then the following conditions are mutually equivalent:

(i) H is order convex.

(ii) $a, c \in H$, $b \in G$, $a \leq b \leq c \Rightarrow b \in H$.

(iii) If Q is an Abelian group and ξ: $G \to Q$ is a group homomorphism with $\ker(\xi) = H$, then Q can be organized into a partially ordered Abelian group with $Q^+ = \xi(G^+)$.

(iv) There is a partially ordered Abelian group Q and a group homomorphism $\xi: G \to Q$ such that $\xi(G^+) \subseteq Q^+$ and $H = \ker(\xi)$.

The intersection of order-convex subgroups of G is again an order-convex subgroup of G.

Definition 3.4. If X is any subset of G, denote by ocs(X) the intersection of all order-convex subgroups of G that contain X. Also, let ssg(X) be the subsemigroup of G consisting of 0 and all sums of finite sequences of elements in X.

Lemma 3.5. Let X be a subset of G^+. Then:
(i) $\text{ocs}(X) = \text{ocs}(\text{ssg}(X)) = \{h \in G \mid \exists y \in \text{ssg}(X), -y \leq h \leq y\}$.
(ii) ocs(X) is a directed subgroup of G.

An element $u \in G^+$ is called an *order unit* iff for every $g \in G$, there exists $n \in \mathbb{Z}^+$ such that $g \leq nu$. By Lemma 3.5, $u \in G^+$ is an order unit iff $\text{ocs}(\{u\}) = G$; hence, if G admits an order unit, then, by Corollary 3.2, G is directed.

The material that follows pertains to the subjects under consideration in this paper and is not explicitly treated in the standard literature on partially ordered algebraic structures.

Definition 3.6. Let $u \in G^+$.
(i) We define the *interval* $G^+[0, u] := \{p \in G \mid 0 \leq p \leq u\}$.
(ii) The element u is *generative* iff $G^+ = \text{ssg}(G^+[0, u])$ and $G = \langle G^+[0, u] \rangle$.

Thus, an element of the positive cone is generative iff it generates the positive cone as a subsemigroup and, in turn, the positive cone generates the group.

Lemma 3.7. If $u \in G^+$ and $G = \langle G^+[0, u] \rangle$, then u is an order unit.

Proof. Suppose $g \in G = \langle G^+[0, u] \rangle$. Then there are elements $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in G^+[0, u]$ such that $g = \Sigma_i a_i - \Sigma_j b_j \leq \Sigma_i a_i \leq nu$. ■

As a consequence of Lemma 3.7, every generative element of G^+ is an order unit. If G is lattice ordered, then every order unit is generative; otherwise, as the following example shows, order units are not necessarily generative.

Example 3.8. If $G = \mathbb{Z}$ as a partially ordered Abelian group with the nonstandard positive cone $G^+ := \{m \in \mathbb{Z} \mid m - 2 \in \mathbb{Z}^+\} \cup \{o\}$, then 2 is an order unit in $\mathbb{Z}$, but it is not generative.

Lemma 3.9. Let $u \in G^+$ be a generative order unit and let H be an order-convex directed subgroup of G. Then

$$H = \langle H \cap G^+[0, u] \rangle$$

Proof. By Lemma 3.1, $H = \langle H \cap G^+ \rangle$, so it will be sufficient to prove that $H \cap G^+ \subseteq \langle H \cap G^+[0, u] \rangle$. Let $h \in H \cap G^+$. Since u is generative, $h \in G^+ = \text{ssg}(G^+[0, u])$ implies that there is a finite sequence $a_1, a_2, \ldots, a_n \in G^+[0, u]$ such that $h = \Sigma_i a_i$. For each $j = 1, 2, \ldots, n$, we have $0 \le a_j \le \Sigma_i a_i = h$ and, owing to the facts that $h \in H$ and H is order convex, we have $a_j \in H \cap G^+[0, u]$; hence, $h \in \langle H \cap G^+[0, u] \rangle$. ∎

4. INTERVAL EFFECT ALGEBRAS

If G is a partially ordered Abelian group and $u \in G^+$, then the interval $G^+[0, u]$ can be organized into an effect algebra with unit u by taking $\oplus$ to be the restriction to $G^+[0, u]$ of $+$ on G. As such, the effect algebra partial order on $G^+[0, u]$ coincides with the restriction to $G^+[0, u]$ of the partial order on G.

Definition 4.1. An effect algebra of the form $G^+[0, u]$, or isomorphic to an effect algebra of this form, is called an *interval effect algebra.*

In Bennett and Foulis (n.d.) it is shown that a sub-effect algebra of an interval effect algebra is again an interval effect algebra. In Foulis *et al.* (1994) it is proved that the class of interval effect algebras is closed under the formation of Cartesian products, horizontal sums, and tensor products.

Example 4.2. The interval effect algebra $\mathbb{R}^+[0, 1]$ is called the *standard scale algebra.* More generally, a totally ordered effect algebra is called a *scale algebra.*

If $\mathcal{H}$ is a one-dimensional Hilbert space, then the standard effect algebra $E(\mathcal{H})$ is isomorphic to the standard scale algebra $\mathbb{R}^+[0, 1]$.

Definition 4.3. A *probability measure* on an effect algebra E is a morphism $\omega: E \to \mathbb{R}^+[0, 1]$ from E to the standard scale algebra.

Since a probability measure ω on an effect algebra E is a morphism, its effect kernel $I := \omega^{-1}(0)$ is an ideal in E. It can be shown that every interval effect algebra admits at least one probability measure and that, conversely, if there are enough probability measures on E to determine the partial order $\le$, then E is an interval effect algebra (Bennett and Foulis, n.d.).

If E is an effect algebra and K is an Abelian group, a mapping $\phi: E \to K$ is called a *K-valued measure* iff, for $p, q \in E$ with $p \perp q$, $\phi(p \oplus q) = \phi(p) + \phi(q)$. A proof of the following theorem can be found in Bennett and Foulis (n.d.).

Theorem 4.4. If E is an interval effect algebra, then there is a partially ordered Abelian group G with a generative order unit u such that $E = G^+[0, u]$

and, for every Abelian group K, every K-valued measure ϕ: $G^+[0, u] \to K$ can be extended uniquely to a group homomorphism ϕ^*: $G \to K$.

The group G in Theorem 4.4, which is unique up to a group isomorphism, is called the *universal group* for the interval effect algebra E.

5. THE GROUP KERNEL OF A MORPHISM OF INTERVAL EFFECT ALGEBRAS

In this section, it will be convenient to adopt the following notation.

Standing Notation 5.1. For the remainder of this section, G is the universal group with unit u for the interval effect algebra $E = G^+[0, u]$, Q is a partially ordered Abelian group with $v \in Q^+$, $F = Q^+[0, v]$ is organized into an interval effect algebra, and ϕ: $E \to F$ is a morphism. Thus, ϕ: $E \to Q$ is a Q-valued measure, so it admits a unique extension to a group homomorphism ϕ^*: $G \to Q$.

Since $G^+ = \text{ssg}(E)$ and $F \subseteq Q^+$, we have $\phi^*(G^+) \subseteq Q^+$, so ϕ^*: $G \to Q$ is order preserving and, by part (iv) of Theorem 3.3, $\ker(\phi^*)$ is an order-convex subgroup of G. Furthermore, ϕ^* is *normalized* in the sense that $\phi^*(u) = v$. In fact, given G and Q as above, it is clear that there is a one-to-one correspondence $\phi \leftrightarrow \phi^*$ between morphisms ϕ: $E \to F$ and normalized order-preserving group homomorphisms ϕ^*: $G \to Q$.

The morphism ϕ: $E \to F$ has the effect kernel $I := \phi^{-1}(0)$. We have to be careful to distinguish between the ideal $I \subseteq E$ and the kernel of the group homomorphism ϕ^*: $G \to Q$. Because ϕ^* is an extension of ϕ, we have $I = \ker(\phi^*) \cap E$.

Definition 5.2. The order-convex subgroup $\ker(\phi^*) \subseteq G$ is called the *group kernel* of ϕ.

In the next two examples and in the remainder of the paper we use the usual notation Z^n for the additive Abelian group obtained by forming the n-fold Cartesian product of Z with itself and we denote by Z_n the additive group of integers modulo n. The *standard positive cone* in Z^n is understood to be

$$(Z^+)^n := \{(z_1, z_2, \ldots, z_n) \mid z_i \in Z^+ \text{ for } i = 1, 2, \ldots, n\}$$

Example 5.3. Let $G := Z^4$ with the nonstandard positive cone

$$G^+ := \{(x, y, z, w) \mid x, y, z, w \in Z^+, w \leq y + z\}$$

and with the unit $u := (1, 1, 1, 1)$. Let $Q := Z^3$ with the standard positive cone $Q^+ := (Z^+)^3$ and with the unit $v := (1, 1, 1)$. The mapping ϕ^*: $G \to$

Q given by $\phi^*(x, y, z, w) := (x, z, w)$ is a group epimorphism with $\phi^*(u) = v$ and $\phi^*(G^+) = Q^+$. The interval effect algebra $E = G^+[0, u]$ is isomorphic to the 12-element orthomodular lattice G_{12} (Kalmbach, 1983, Figure 9.4) and G is its universal group. The interval effect algebra $F = Q^+[0, v]$ is isomorphic to the eight-element Boolean algebra 2^3 and Q is its universal group. The restriction $\phi\colon E \to F$ of ϕ^* to E is a surjective morphism of effect algebras. The effect kernel of ϕ is $I := \phi^{-1}((0, 0, 0)) = \{(0, 0, 0, 0), (0, 1, 0, 0)\}$ and its group kernel is $\ker(\phi^*) = \{(0, y, 0, 0) \mid y \in \mathbb{Z}\}$.

Example 5.4. The *Wright triangle* $E = W_{14} := G^+[0, u]$ (Foulis *et al.*, 1992, Example 2.13) is a 14-element orthoalgebra with universal group $G = \mathbb{Z}^4$, nonstandard positive cone

$$G^+ = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{Z}^+,\ w \leq x + y + z\}$$

and unit $u = (1, 1, 1, 1)$. Let $Q := \mathbb{Z}^3 \times \mathbb{Z}_2$ and define a group epimorphism $\phi^*\colon G \to Q$ for $(x, y, z, w) \in G$ by

$$\phi^*(x, y, z, w) := (x, y, z, \alpha), \qquad \text{where} \quad \alpha \equiv z + w \pmod 2)$$

Then $\ker(\phi^*) \cap G^+ = \{(0, 0, 0, 0)\}$, so, by Theorem 3.3, Q can be organized into a partially ordered Abelian group with positive cone $Q^+ := \phi^*(G^+)$. Let $v := (1, 1, 1, 0) = \phi^*(u) \in Q$ and let $F := Q^+[(0, 0, 0, 0), v]$. Then F is a 14-element orthoalgebra with Q as its universal group, the restriction ϕ of ϕ^* to E is a bijective morphism $\phi\colon E \to F$, the effect kernel of ϕ is $I = \{(0, 0, 0, 0)\} \subseteq E$, but ϕ is not even an effect algebra homomorphism, let alone an isomorphism.

As Example 5.4 illustrates, the articulation between morphisms and kernels that obtains for groups, rings, Boolean algebras, orthomodular lattices, and so on may fail in the category of interval effect algebras. It is the group kernel, not the effect kernel, that plays the crucial role for morphisms of interval effect algebras. This observation suggests the following definition.

Definition 5.5. A morphism $\phi\colon E \to F$ is said to be *regular* iff its group kernel, $\ker(\phi^*)$, coincides with the subgroup $\langle I \rangle$ generated by its effect kernel $I = \phi^{-1}(0)$.

Lemma 5.6. A morphism $\phi\colon E \to F$ is regular iff its group kernel, $\ker(\phi^*)$, is a directed subgroup of G.

Proof. Let $I = \phi^{-1}(0)$. If $\ker(\phi^*) = \langle I \rangle$, then $\ker(\phi^*)$ is directed by Lemma 3.1. Conversely, suppose $\ker(\phi^*)$ is directed. Since ϕ^* is an extension of ϕ, we have $I = \ker(\phi^*) \cap E$, so $\ker(\phi^*) = \langle I \rangle$ by Lemma 3.9. ■

The morphism in Example 5.3 is regular, but the morphism in Example 5.4 is not.

6. REGULAR QUOTIENTS OF INTERVAL EFFECT ALGEBRAS

According to the following theorem, an ideal I in an interval effect algebra E induces a regular morphism in a natural way.

Theorem 6.1. Suppose G is the universal group for the interval effect algebra $E = G^+[0, u]$ and let I be an ideal in E. Let $\phi^*: G \to Q$ be the natural group epimorphism with kernel ocs(I) onto the quotient group $Q = G/\text{ocs}(I)$, organize Q into a partially ordered Abelian group with positive cone $Q^+ := \phi^*(G^+)$, let $v := \phi^*(u)$, and define $F := Q^+[0, v]$ to be the corresponding interval effect algebra. Then v is a generative order unit in Q and the restriction ϕ of ϕ^* to E is a regular morphism $\phi: E \to F$ with effect kernel $\phi^{-1}(0) = \text{ocs}(I) \cap E$.

Proof. By part (iii) of Theorem 3.3, $Q^+ = \phi^*(G^+)$ is a cone in Q. Since u is a generative order unit in G and $\phi^*: G \to Q$ is an order-preserving epimorphism, it follows that $v = \phi^*(u)$ is a generative order unit in Q. Clearly, $\phi: E \to F$ is a morphism with effect kernel $\phi^{-1}(0) = \ker(\phi^*) \cap E = \text{ocs}(I) \cap E$. By part (ii) of Lemma 3.5, ocs(I) is a directed subgroup of G, so $\phi: E \to F$ is regular by Lemma 5.6. ■

Definition 6.2. Let G be the universal group for the interval effect algebra $E = G^+[0, u]$ and let I be an ideal in E. Then the interval effect algebra $Q^+[0, v]$ in Theorem 6.1 is called the *regular quotient effect algebra of E modulo I* (in the category of interval effect algebras), and is written as $E/\text{ocs}(I)$. The morphism $\phi: E \to E/\text{ocs}(I)$ in Theorem 6.1 is called the *regular quotient morphism*.

Suppose G is the universal group for $E = G^+[0, u]$, I is an ideal in E, and $\phi^*: G \to Q$ is a group epimorphism with $\ker(\phi^*) = \text{ocs}(I)$. Then Q can be organized into a partially ordered Abelian group with $Q^+ := \phi^*(G^+)$, Q is isomorphic to the quotient group $G/\text{ocs}(I)$, and $Q^+[0, \phi^*(u)]$ is isomorphic to the regular quotient $E/\text{ocs}(I)$. Therefore, when convenient, we shall indulge in a slight abuse of notation and terminology by identifying $E/\text{ocs}(I)$ with $Q^+[0, \phi^*(u)]$ and referring to ϕ as the regular quotient morphism.

As a first indication that Definition 6.2 is consistent with our usual understanding of quotients, we submit the following result, which, roughly speaking, says that the regular quotient $(E_1 \times E_2)/\text{ocs}(E_1)$ is E_2.

Theorem 6.3. If E_1 and E_2 are interval effect algebras, $E = E_1 \times E_2$ is the effect-algebra Cartesian product, and $I = E_1 \times \{0\}$, then $E/\text{ocs}(I)$ is isomorphic to E_2.

Proof. Let $E_i = G_i^+[0, u_i]$, where G_i is the universal group for E_i, $i = 1, 2$. By Foulis *et al.* (1994), $G := G_1 \times G_2$, with the positive cone $G^+ :=$

$G_1^+ \times G_2^+$ and with generative order unit $u := (u_1, u_2)$, is the universal group for $E_1 \times E_2$. Thus, $\langle I \rangle = \{(g_1, 0) | g_1 \in G_1\}$ is an order convex subgroup of G, so $\langle I \rangle = \text{ocs}(I)$. The projection mapping ϕ^*: $G \to G_2$ given by $\phi^*(g_1, g_2) := g_2$ for all $(g_1, g_2) \in G$ is a group epimorphism with $\ker(\phi^*) = \langle I \rangle = \text{ocs}(I)$ and $\phi^*(G^+) = G_2^+$, so $(E_1 \times E_2)/\text{ocs}(I)$ is isomorphic to $G_2^+[0, u_2] = E_2$. ■

Definition 6.4. Let G be the universal group for $E = G^+[0, u]$ and let I be an ideal in E.

(i) I is *closed* iff $I = \text{ocs}(I) \cap E$.

(ii) The *closure* of the ideal I is $I^* := \text{ocs}(I) \cap E$. Thus, I is closed iff $I = I^*$.

Lemma 6.5. Let G be the universal group for $E = G^+[0,]$, let I be an ideal in E, and let ϕ: $E \to E/\text{ocs}(I)$ be the regular quotient morphism.

(i) $I \subseteq I^* = \ker(\phi^*) \cap E$.

(ii) $\text{ocs}(I) = \text{ocs}(I^*)$.

(iii) I^* is a closed ideal in E.

(iv) $E/\text{ocs}(I) = E/\text{ocs}(I^*)$.

(v) I is closed iff $y \in \text{ssg}(I), p \in E, p \leq y \Rightarrow p \in I$.

(vi) If I is closed, then $\langle I \rangle = \text{ocs}(I)$.

Proof. Part (i) is obvious. To prove part (ii), note that $\text{ocs}(I) \subseteq \text{ocs}(I^*)$ follows from the fact that $I \subseteq I^*$. Conversely, $I^* = \text{ocs}(I) \cap E \subseteq \text{ocs}(I)$, so $\text{ocs}(I^*) \subseteq \text{ocs}(I)$.

Because $\text{ocs}(I)$ is a subgroup of G, it is clear that $I^* = \text{ocs}(I) \cap E$ is closed under $\oplus$. If $p \in E$ and $0 \leq p \leq q \in \text{ocs}(I) \cap E$, then, owing to the fact that $\text{ocs}(I)$ is order convex, $p \in \text{ocs}(I) \cap E$. Therefore $\text{ocs}(I) \cap E$ is an ideal in E. By part (ii), $I^* = \text{ocs}(I^*) \cap E$, so I^* is a closed ideal, and the proof of part (iii) is complete.

Part (iv) is an obvious consequence of parts (ii) and (iii) and part (v) follows directly from part (i) of Lemma 3.5. To prove part (vi), suppose that $I = \text{ocs}(I) \cap E$. Then, by Lemma 3.9, $\text{ocs}(I) = \langle \text{ocs}(I) \cap E \rangle = \langle I \rangle$. ■

Clearly, the closed ideals in an interval effect algebra E are precisely the kernels of regular morphisms on E. We suspect that the converse of part (vi) of Lemma 6.5 fails, but we do not know of an example to show this.

Example 6.6. Let Z_2 be the additive group of integers modulo 2 and let $G := Z \times (Z_2)^3$. Define a cone G^+ in G as follows: $(x, \alpha, \beta, \gamma) \in G^+$ iff $x > 1$, or $x = 1$ and one of α, β, γ is nonzero, or $x = 0$ and $\alpha = \beta = \gamma = 0$. Then $u := (3, 0, 0, 0)$ is a generative order unit in G and G is the universal group for the interval effect algebra Fa_{16}, called the *Fano* effect algebra. [The atoms in Fa_{16} can be identified in a natural way with the points in the

Fano projective plane (Bennett and Foulis, 1993, Section 7.] Let $I := \{(0, 0, 0, 0), (1, 1, 1, 1)\}$. Then $\langle I \rangle = \{(n, n, n, n) \mid n \in \mathbb{Z}\}$, where the last three components are understood to be reduced modulo 2. Note that for the given partial order on G,

$$(0, 0, 0, 0) \leq (1, 1, 0, 0) \leq (2, 0, 0, 0) \in \langle I \rangle$$

but $(1, 1, 0, 0) \notin \langle I \rangle$, so $\langle I \rangle \neq \text{ocs}(I)$. Indeed, if $(x, \alpha, \beta, \gamma) \in G$, let $n \in \mathbb{Z}^+$ with $1 + |x| \leq n$, and note that

$$-(n, n, n, n) \leq (x, \alpha, \beta\ \gamma) \leq (n, n, n, n)$$

so $\text{ocs}(I) = G$ by part (i) of Lemma 3.5. Consequently, I is not a closed ideal in Fa_{16}, and in fact $I^* = Fa_{16}$.

Lemma 6.7. Let a be a nonisotropic atom in an interval algebra E with universal group G. Then $I := \{0, a\}$ is an ideal in E, $\langle I \rangle = \langle a \rangle$, and, if $\langle a \rangle$ is order-convex, then I is a closed ideal in E.

Proof. That I is an ideal and $\langle I \rangle = \langle a \rangle$ is clear. Assume $\langle a \rangle = \langle I \rangle$ is order-convex so that $\langle a \rangle = \text{ocs}(I)$. Suppose I fails to be closed. Then there is an element $p \in \text{ocs}(I) \cap E = \langle a \rangle \cap E$ such that $p \notin I$. Since $p \in \langle a \rangle$, there exists $n \in \mathbb{Z}$ such that $0 \leq p = na \leq u$. Because $0, a \in I$, it follows that $n \neq 0, 1$. Also, since $a, na \in G^+$, we cannot have $n \leq 0$. Therefore, $n \geq 2$, so $2a \leq u$, contradicting the fact that a is nonisotropic. ∎

In Examples 5.3 and 6.6 above and Examples 9.1, 10.1, and 10.2 below, the ideal I has the form $I = \{0, a\}$, where a is a nonisotropic atom in the given interval effect algebra E. Only in Example 6.6 does the cyclic subgroup $\langle I \rangle = \langle a \rangle$ of G fail to be order-convex. Therefore, in all of these cases except for Example 6.6, the ideal $I = \{0, a\}$ is closed by Lemma 6.7.

By part (iv) of Lemma 6.5, in forming regular quotients of an interval effect algebra E by ideals I in E, we only need to consider closed ideals. As is easily verified, the closed ideals in E form a complete lattice under $\subseteq$ which is isomorphic to the complete lattice of all directed and order-convex subgroups of the universal group G of E. The closed ideals I in E that are the kernels of surjective regular morphisms $\phi: E \to F$ are the analogues of normal subgroups in group theory. (See Section 9 below, where we address the question of surjectivity.)

7. THE BOOLEAN CASE

In this section we sketch a proof showing that Definition 6.2 is consistent with the usual definition of a quotient of a Boolean algebra by a Boolean ideal. Thus, for the remainder of the section, *we assume that X is a compact Hausdorff totally disconnected topological space.*

We regard $\mathbb{Z}^X := \{f | f: X \to \mathbb{Z}\}$ as an additive group under pointwise operations. The Abelian group $\mathbb{Z}^X$ is understood to be partially ordered by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, with the corresponding positive cone $(\mathbb{Z}^+)^X$. We define $G(X)$ to be the subgroup of $\mathbb{Z}^X$ consisting of all functions $f: X \to \mathbb{Z}$ that are continuous when $\mathbb{Z}$ is given the discrete topology. The Abelian group $G(X)$ is partially ordered by the induced positive cone $G(X)^+ := G(X) \cap (\mathbb{Z}^+)^X$.

Because X is compact, a function $f \in \mathbb{Z}^X$ belongs to $G(X)$ iff it can be written in the canonical form

$$f = \sum_{i=1}^{n} a_i \chi_{C_i}$$

where χ_{C_i} is the characteristic set function of $C_i \subseteq X$, each C_i is a nonempty compact open subset of X, the pairwise disjoint sets C_i, $i = 1, 2, \ldots, n$, form a partition of X, and $a_1, a_2, \ldots, a_n$ are distinct integers. Elements of the interval effect algebra $E(X) := G(X)^+[0, \chi_X]$ are the characteristic set functions of compact open subsets of X; hence, $E(X)$ is a Boolean algebra. By the Stone representation theorem (Stone, 1936), every Boolean algebra can be represented as an $E(X)$, where X is uniquely determined up to a homeomorphism.

Evidently, χ_X is a generative order unit in $G(X)$. Furthermore, $G(X)$ is the universal group for $E(X)$. Indeed if K is an Abelian group and $\phi: E(X) \to K$ is a K-valued measure, then ϕ can be extended to a group homomorphism $\phi^*: G(X) \to K$ by defining

$$\phi^*(f) := \sum_{i=1}^{n} a_i \phi(\chi_{C_i})$$

where $f \in G(X)$ is expressed in canonical form.

Let U be an open subset of X and let $I(U)$ be the subset of $E(X)$ consisting of all characteristic set functions χ_C of compact open sets $C \subseteq U$. Clearly, $I(U)$ is an ideal in $E(X)$. Note that, for a Boolean algebra, *ideals in the effect-algebra sense coincide with Boolean ideals in the usual sense.*

By a standard argument, every ideal in $E(X)$ has the form $I(U)$ for a uniquely determined open set $U \subseteq X$. Let $Y := X \setminus U$ be the complement of U in X, noting that Y is a compact Hausdorff totally disconnected space under the relative topology inherited from X. Define $\phi: E(X) \to E(Y)$ by $\phi(\chi_C) := \chi_{C \cap Y}$ for all compact open subsets C of X. Obviously, ϕ is a Boolean homomorphism with kernel $I(U)$. Using the fact that X admits a basis of compact open sets, a straightforward argument shows that $\phi: E(X) \to E(Y)$ is surjective. Therefore, ϕ induces a Boolean isomorphism of the usual Boolean quotient algebra $E(X)/I(U)$ onto the Boolean algebra $E(Y)$.

Now $\phi: E(X) \to E(Y) \subseteq G(Y)$, ϕ is a $G(Y)$-valued measure, and $G(X)$ is the universal group of $E(X)$, so there is a uniquely defined group homomorphism $\phi^*: G(X) \to G(Y)$ that agrees with ϕ on $E(X)$. Since $E(Y)$ generates $G(Y)$ and $\phi(E(X)) = E(Y)$, it follows that $\phi^*: G(X) \to G(Y)$ is surjective. Also, it is clear that $\phi^*(G(X)^+) = G(Y)^+$. Using the canonical form of elements in $G(X)$, we verify that

$$\ker(\phi^*) = \{f \in G(X) \mid f(Y) = 0\} = \langle I(U) \rangle$$

and it follows that $\ker(\phi^*) = \mathrm{ocs}(I(U))$. Therefore the regular quotient effect algebra $E(X)/\mathrm{ocs}(I(U))$ can be identified with

$$G(Y)^+[0, \phi^*(\chi_X)] = G(Y)^+[0, \chi_Y] = E(Y)$$

which in turn is isomorphic to the corresponding quotient Boolean algebra $E(X)/I(U)$.

8. QUOTIENTS OF STANDARD EFFECT ALGEBRAS

In this section let $\mathcal{H}$ be a Hilbert space, let $\mathrm{B}(\mathcal{H})$ be the *-algebra of all bounded operators on $\mathcal{H}$, and let $\mathcal{G}(\mathcal{H})$ be the group under addition of all self-adjoint operators $A = A^* \in \mathrm{B}(\mathcal{H})$. We organize $\mathcal{G}(\mathcal{H})$ into a partially ordered Abelian group with positive cone

$$\mathcal{G}^+(\mathcal{H}) := \{A^2 \mid A \in \mathcal{G}(\mathcal{H})\} = \{BB^* \mid B \in \mathrm{B}(\mathcal{H})\}$$

and consider the standard effect algebra $\mathrm{E}(\mathcal{H}) := \mathcal{G}^+(\mathcal{H})[0, 1]$. By Corollary 4.7 of Bennett and Foulis (n.d.), $\mathcal{G}(\mathcal{H})$ is the universal group for $\mathrm{E}(\mathcal{H})$. Proofs of most of the observations below follow either from Section 6 of Greechie *et al.* (1995) or from standard operator-theoretic arguments.

Denote by $\mathrm{P}(\mathcal{H})$ the set of all *projection operators* $P = P^2 \in \mathcal{G}(\mathcal{H})$, noting that $\mathrm{P}(\mathcal{H})$ is a sub-effect algebra of $\mathrm{E}(\mathcal{H})$ and that, as such, it forms a complete orthomodular lattice. If $P \in \mathrm{E}(\mathcal{H})$, the order ideal $\mathcal{G}^+[0, P] := \{A \in \mathcal{G}(\mathcal{H}) \mid 0 \leq A \leq P\}$ is an effect ideal in $\mathrm{E}(\mathcal{H})$ if and only if $P \in \mathrm{P}(\mathcal{H})$. If $P \in \mathrm{P}(\mathcal{H})$, then

$$\mathcal{G}^+[0, P] = \{A \in \mathrm{E}(\mathcal{H}) \mid A = AP = PA\}$$

and

$$\mathcal{G}^+[0, P'] = \{A \in \mathrm{E}(\mathcal{H}) \mid AP = PA = 0\}$$

Furthermore,

$$\langle \mathcal{G}^+[0, P'] \rangle = \mathrm{ocs}(\mathcal{G}^+[0, P']) = \{A \in \mathcal{G}(\mathcal{H}) \mid AP = PA = 0\}$$

so $\mathcal{G}^+[0, P']$ is a closed ideal in $\mathrm{E}(\mathcal{H})$.

Let $P \in \mathrm{P}(\mathcal{H})$ with $P \neq 0$, 1 and let $\mathcal{M}$ be the closed linear subspace of $\mathcal{H}$ given by $\mathcal{M} := P(\mathcal{H})$. There is a rather natural order-preserving group epimorphism $(\Phi_P)^*\colon \mathcal{G}(\mathcal{H}) \to \mathcal{G}(\mathcal{M})$ given by $((\Phi_P)^*A)\alpha := PA\alpha$ for all $\alpha \in \mathcal{M}$, and we have

$$\ker((\Phi_P)^*) = \{A \in \mathcal{G}(\mathcal{H}) \mid PAP = 0\}$$

Of course, $\ker((\Phi_P)^*)$ is order-convex, but it is *not directed.* The restriction $\Phi_P\colon \mathrm{E}(\mathcal{H}) \to \mathrm{E}(\mathcal{M})$ of $(\Phi_P)^*$ to $\mathrm{E}(\mathcal{H})$ is a surjective effect morphism with effect kernel

$$(\Phi_P)^{-1}(0) = \mathcal{G}^+[0, P'] = \{A \in \mathrm{E}(\mathcal{H}) \mid AP = PA = 0\}$$

However, in spite of the fact that $\Phi_P\colon \mathrm{E}(\mathcal{H}) \to \mathrm{E}(\mathcal{M})$ is an important effect algebra morphism from the point of view of quantum logic, it is *not regular.* In fact, the directed order-convex subgroup

$$\langle(\Phi_P)^{-1}(0)\rangle = \langle\mathcal{G}^+[0, P']\rangle = \{A \in \mathcal{G}(\mathcal{H}) \mid AP = PA = 0\}$$

is a *proper* subgroup of $\ker((\Phi_P)^*) = \{A \in \mathcal{G}(\mathcal{H}) \mid PAP = 0\}$.

With the notation of the last paragraph, let $\mathcal{C}$ be the additive subgroup of $\mathrm{B}(\mathcal{H})$ given by

$$\mathcal{C} := \{C \in \mathrm{B}(\mathcal{H}) \mid P'C = CP = 0\}$$

and let $\mathcal{Q} = \mathcal{G}(\mathcal{M}) \times \mathcal{C}$. Define the group epimorphism $(\phi_P)^*\colon \mathcal{G}(\mathcal{H}) \to \mathcal{Q}$ by $(\phi_P)^*A = ((\Phi_P)^*A, PAP')$ for all $A \in \mathcal{G}(\mathcal{H})$. Then

$$\ker((\phi_P)^*) = \{A \in \mathcal{G}(\mathcal{H}) \mid AP = PA = 0\} = \langle\mathcal{G}^+[0, P']\rangle$$

is a directed order-convex subgroup of $\mathcal{G}^+(\mathcal{H})$, so we can organize $\mathcal{Q}$ into a partially ordered Abelian group with $\mathcal{Q}^+ := \phi^*(\mathcal{G}^+(\mathcal{H}))$. The restriction ϕ_P of $(\phi_P)^*$ to $\mathrm{E}(\mathcal{H})$ is a regular surjective effect-algebra morphism $\phi_P\colon \mathrm{E}(\mathcal{H}) \to \mathcal{Q}^+[(0_P, 0), (1_P, 0)]$, where 0_P and 1_P are the zero and identity operators in $\mathrm{B}(\mathcal{M})$. Thus, the regular quotient $\mathrm{E}(\mathcal{H})/\mathrm{ocs}(\mathcal{G}^+[0, P'])$ is not isomorphic to $\mathrm{E}(\mathcal{M})$ as one might expect, but rather to an interval in $\mathcal{Q} = \mathcal{G}(\mathcal{M}) \times \mathcal{C}$.

A *density operator* on $\mathcal{H}$ is a trace-class operator $W \in \mathcal{G}^+(\mathcal{H})$ with $\mathrm{tr}(W) = 1$. Such an operator determines a probability measure $\omega_W\colon \mathrm{E}(\mathcal{H}) \to \mathbb{R}^+[0, 1]$ according to $\omega_W(A) := \mathrm{tr}(WA)$ for all $A \in \mathrm{E}(\mathcal{H})$, and the unique extension of ω_W to a group homomorphism $(\omega_W)^*\colon \mathcal{G}(\mathcal{H}) \to \mathbb{R}$ is given by $(\omega_W)^*(A) := \mathrm{tr}(WA)$ for all $A \in \mathcal{G}(\mathcal{H})$. If $P \in \mathrm{P}(\mathcal{H})$ is the projection onto the orthogonal complement of the null space of W, then the effect kernel of ω_W is

$$(\omega_W)^{-1}(0) = \{A \in \mathrm{E}(\mathcal{H}) \mid AP = PA = 0\} = \mathcal{G}^+[0, P']$$

However, in general, ω_W *is not a regular morphism* because $\mathrm{ocs}(\mathcal{G}^+[0, P']) = \langle\mathcal{G}^+[0, P']\rangle$ is not equal to the group kernel $\ker((\omega_W)^*)$.

In summary, if W is a density operator on $\mathcal{H}$ and P is the projection onto the orthogonal complement $\mathcal{M}$ of the null space of W, then there are three surjective effect-algebra morphisms ϕ_P: $E(\mathcal{H}) \to E(\mathcal{H})/\text{ocs}(\mathcal{G}^+[0, P'])$, Φ_P: $E(\mathcal{H}) \to E(\mathcal{M})$, and ω_W: $E(\mathcal{H}) \to \mathbf{R}^+[0, 1]$, all with the same effect kernel $\{A \in E(\mathcal{H}) \mid AP = PA = 0\}$, but with three group kernels

$$\ker((\phi_P)^*) \subseteq \ker((\Phi_P)^*) \subseteq \ker((\omega_W)^*)$$

which, in general, are all different. [If W represents a *pure state*, so that $W = P$, then $\ker((\Phi_P)^*) = \ker((\omega_W)^*)$.] All of this will have to be reconciled in any general theory of quotients of effect algebras.

9. SURJECTIVITY

The traditional mathematical notion of a quotient leads one to expect that a regular quotient morphism ϕ: $E \to E/\text{ocs}(I)$ ought to be surjective. The following example shows that, for interval effect algebras, this expectation is not necessarily fulfilled.

Example 9.1. Let $G := \mathbf{Z}^5$, with the positive cone

$$G^+ := \{(x, y, z, p, q) \in (\mathbf{Z}^+)^5 \mid q \leq x + y + z + p\}$$

and with the generative order unit $u := (1, 1, 1, 1, 2)$. There are eight atoms and 36 elements in the effect algebra $FL_{36} := G^+[0, u]$, called the *Frazier–Lock cube*. In fact, FL_{36} is an orthoalgebra. Let $I := \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0)\}$, noting that I is an ideal in FL_{36}, and $\langle I \rangle = \{(n, 0, 0, 0, 0) \mid n \in \mathbf{Z}\}$ is an order-convex subgroup of G. The mapping ϕ^*: $\mathbf{Z}^5 \to \mathbf{Z}^4$ given by $\phi^*(x, y, z, p, q) := (y, z, p, q)$ is a group epimorphism with $\ker(\phi^*) = \langle I \rangle = \text{ocs}(I)$, and $\phi^*(G^+) = (\mathbf{Z}^+)^4$, the standard positive cone in $\mathbf{Z}^4$. Therefore, $FL_{36}/\text{ocs}(I)$ can be identified with $(\mathbf{Z}^+)^4[(0, 0, 0, 0), (1, 1, 1, 2)]$, which is a 24-element effect algebra isomorphic to the Cartesian product of the Boolean algebra 2^3 and a chain with three elements. With the notation of Foulis *et al.* (1994, Example 4.3), this is the *rectangular trellis* $RT(1, 1, 1, 2)$. The image of FL_{36} under the regular morphism ϕ consists of all elements in $FL_{36}/\text{ocs}(I)$ *with the exception of* $(1, 1, 1, 0)$ *and its orthosupplement* $(0, 0, 0, 2)$.

There seems to be no reasonable way to repair the lack of surjectivity of the regular morphism ϕ in Example 9.1. For instance, one could propose that the true quotient of FL_{36} by I ought to be the *image* $\phi(FL_{36})$ of FL_{36} in $\mathbf{Z}^4$ under ϕ, but it turns out that $\phi(FL_{36})$ is not even a sub-effect algebra of the interval effect algebra $(\mathbf{Z}^+)^4[(0, 0, 0, 0), (1, 1, 1, 2)]$. In fact, there is no interval effect algebra that is an image of FL_{36} under a morphism with I as its effect kernel. Evidently the difficulty is simply that, even though I is closed, it is not a "well-behaved" ideal.

For surjective regular quotient morphisms, we have the following theorem.

Theorem 9.2. Let G be the universal group of $E = G^+[0, u]$, let Q be a partially ordered Abelian group, let v be a generative order unit in Q^+, and let $F = Q^+[0, v]$. Then, if $\phi: E \to F$ is a surjective regular morphism, it follows that Q is the universal group for F.

Proof. Let $\phi^*: G \to Q$ be the unique extension of $\phi: E \to F$ to a group homomorphism. Suppose K is an Abelian group and that $\kappa: F \to K$ is a K-valued measure. Then $\kappa \circ \phi: E \to K$ is also a K-valued measure, so it can be extended to a group homomorphism $(\kappa \circ \phi)^*: G \to K$. Evidently,

$$(\kappa \circ \phi)^*(\ker(\phi^*) \cap E) = \kappa(\phi(\ker(\phi^*) \cap E)) = \{0\}$$

whence $\langle \ker(\phi^*) \cap E \rangle \subseteq \ker((\phi \circ \kappa)^*)$. Since $\phi: E \to F$ is a regular morphism,

$$\ker(\phi^*) = \langle \phi^{-1}(0) \rangle = \langle \ker(\phi^*) \cap E \rangle \rangle \subseteq \ker((\phi \circ \kappa)^*)$$

and it follows that there is a group homomorphism $\kappa^*: Q \to K$ such that $\kappa^* \circ \phi^* = (\kappa \circ \phi)^*$. Suppose $q \in F$. Since $\phi: E \to F$ is surjective, there exists $p \in E$ with $q = \phi(p)$ and it follows that

$$\kappa^*(q) = \kappa^*(\phi(p)) = \kappa^*(\phi^*(p)) = (\kappa \circ \phi)^*(p) = (\kappa \circ \phi)(p) = \kappa(q).$$

Therefore, the group homomorphism κ^* is an extension of κ. ∎

In Example 9.1, it turns out that, in spite of the fact that the regular morphism ϕ is not surjective, the group $\mathbb{Z}^4$ with the standard positive cone $(\mathbb{Z}^+)^4$ and the order unit (1, 1, 1, 2) is the universal group for $FL_{36}/\text{ocs}(I)$. As a matter of fact, we do not know of an example in which the conditions in Theorem 6.1 hold, but Q is not the universal group of F.

10. QUOTIENTS OF ORTHOALGEBRAS AND ORTHOMODULAR LATTICES

There are orthoalgebras that are not interval effect algebras, but they are the exception rather than the rule, and they are not of great interest in quantum logic because they never have a "full set of states" (Bennett and Foulis, n.d.). The following example shows that the regular quotient of an interval orthoalgebra by an ideal need not be an orthoalgebra. Thus, *the class of interval orthoalgebras is not closed under the formation of regular quotients,* a fact that provides further justification for the study of effect algebras that are more general than orthoalgebras.

Example 10.1. As in Example 5.4, let $W_{14} = G^+[0, u]$ be the Wright triangle with universal group $G = \mathbb{Z}^4$, positive cone

$$G^+ = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{Z}^+, w \leq x + y + z\}$$

and unit $u = (1, 1, 1, 1)$. Let $I := \{(0, 0, 0, 0), (0, 1, 0, 1)\}$, noting that I is an ideal in W_{14}. A calculation shows that

$$\text{ocs}(I) = \langle I \rangle = \{(0, n, 0, n) \mid n \in \mathbb{Z}\}$$

and that the mapping $\phi^*: G \to \mathbb{Z}^3$ defined for $(x, y, z, w) \in G$ by

$$\phi^*(x, y, z, w) := (x, z, x + y + z - w)$$

is a group epimorphism with kernel $\langle I \rangle$. Evidently, $\phi^*(G^+) = (\mathbb{Z}^+)^3$ is the standard positive cone in $\mathbb{Z}^3$, and $v := \phi^*(u) = (1, 1, 2)$. Thus, $W_{14}/\text{ocs}(I)$ is isomorphic to $(\mathbb{Z}^+)^3[0, v]$, the 12-element effect algebra obtained by taking the Cartesian product of the Boolean algebra 2^2 with the three-element chain $\mathbb{Z}^+[0, 2]$. Again with the notation of Foulis *et al.* (1994), this is the *rectangular trellis* $RT(1, 1, 2)$. The element $(0, 0, 1) \in (\mathbb{Z}^+)^3[0, v]$ is isotropic, so $W_{14}/\text{ocs}(I)$ is not an orthoalgebra.

In the class of orthomodular lattices, there is already a well-developed theory of quotients. Indeed, if L is an OML, then a lattice ideal I in L is called a *p-ideal* iff it is closed under perspectivity. For such an ideal, the quotient L/I, defined in a natural way, is again an OML (Kalmbach, 1983). We do not know of an example in which factoring a p-ideal I from an interval orthomodular lattice L produces anything other than $L/\text{ocs}(I)$ as given by Definition 6.2. However, in an OML there are generally lots of effect algebra ideals that are not p-ideals, and we are now in a position to factor them out, too.

Example 10.2. Let $G = \mathbb{Z}^4$ with the nonstandard positive cone

$$G^+ = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{Z}^+, w \leq y + z\}$$

and generative order unit $u = (1, 1, 1, 1)$, and let $G_{12} = G^+[0, u]$ as in Example 5.3. Let $I := \{(0, 0, 0, 0), (0, 1, 0, 0)\}$, noting that I is a lattice ideal, but not a p-ideal, in the OML G_{12}. Also, $\text{ocs}(I) = \langle I \rangle = \{(0, n, 0, 0) \mid n \in \mathbb{Z}\}$ and the mapping $\phi^*: G \to \mathbb{Z}^3$ given by $\phi^*(x, y, z, w) := (x, z, w)$ is a group epimorphism with kernel $\langle I \rangle$. Evidently, $\phi^*(G^+) = (\mathbb{Z}^+)^3$ and $\phi^*(u) = (1, 1, 1)$. Thus, $G_{12}/\text{ocs}(I)$ is the eight-element Boolean algebra 2^3. Here, both G_{12} and $G_{12}/\text{ocs}(I)$ are OMLs, and the regular quotient morphism ϕ: $G_{12} \to G_{12}/\text{ocs}(I)$ is surjective, but it is *not* an OML homomorphism. (The kernel of an OML homomorphism is always a p-ideal.) In Foulis and Bennett (1994), it is shown that an effect-algebra morphism between OMLs is a homomorphism of OMLs iff it is an effect-algebra homomorphism. Indeed, ϕ is not an effect-algebra homomorphism because, for instance, the elements

(0, 0, 1, 1) and (0, 1, 0, 1) are disjoint in G_{12}, but their images (0, 1, 1) and (0, 0, 1) under ϕ are not disjoint in 2^3.

REFERENCES

Bennett, M. K., and Foulis, D. J. (1993). Tensor products of orthoalgebras, *Order*, **10**(3), 271–282.

Bennett, M. K., and Foulis, D. J. (n.d.). Interval and scale effect algebras, *Advances in Mathematics*, to appear.

Dalla Chiara, M. L., and Giuntini, R. (1989). Paraconsistent quantum logics, *Foundations of Physics*, **19**(7), 891–904.

Dvurečenskij, A. (1995). Tensor product of difference posets, *Transactions of the American Mathematical Society*, **347**(3), 1043–1057.

Dvurečenskij, A., and Pulmannová, S. (1994). Tensor products of D-posets and D-test spaces, *Reports on Mathematical Physics*, **34**(3), 251–275.

Foulis, D. J., and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics*, **24**(10), 1331–1352.

Foulis, D. J., Greechie, R. J., and Rüttimann, G. T. (1992). Filters and supports in orthoalgebras, *International Journal of Theoretical Physics*, **31**(5), 789–802.

Foulis, D. J., Bennett, M. K., and Greechie, R. J. (1994). Sums and products of interval algebras, *International Journal of Theoretical Physics*, **33**(11), 2119–2136.

Foulis, D. J., Bennett, M. K., and Greechie, R. J. (1996). Test groups, *International Journal of Theoretical Physics*, to appear.

Giuntini, R., and Greuling, H. (1989). Toward a formal language for unsharp properties, *Foundations of Physics*, **19**(7), 931–945.

Goodearl, K. R. (1986). *Partially Ordered Abelian Groups with Interpolation*, American Mathematical Society, Providence, Rhode Island.

Greechie, R. J., and Foulis, D. J. (1995). The transition to effect algebras, *International Journal of Theoretical Physics*, **34**(8), 1369–1382.

Greechie, R. J., Foulis, D. J., and Pulmannová, S. (1995). The center of an effect algebra, *Order*, **12**, 91–106.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.

Kläy, M., Randall, C. H., and Foulis, D. J. (1987). Tensor products and probability weights, *International Journal of Theoretical Physics*, **26**(3), 199–219.

Kôpka, F. (1992). D-Posets of fuzzy sets, *Tatra Mountains Mathematical Publications*, **1**, 83–87.

Kôpka, F., and Chovanec, F. (1994). D-Posets, *Mathematica Slovaca*, **44**(1), 21–33.

Mesiar, R. (1993). Fuzzy logics and observables, *International Journal of Theoretical Physics*, **32**(7), 1143–1151.

Navara, M., and Pták, P. (1993). Difference posets and orthoalgebras, *Department of Mathematics Report Series, Czech Technical University in Prague, Faculty of Electrical Engineering*, No. 93-8, pp. 1–5.

Pulmannová, S. (1985). Tensor product of quantum logics, *Journal of Mathematical Physics*, **26**(1), 1–5.

Randall, C. H., and Foulis, D. J. (1981). Operational statistics and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., Bibliographisches Institut Mannheim, Germany.

Schroeck, F. E., Jr., and Foulis, D. J. (1990). Stochastic quantum mechanics viewed from the language of manuals, *Foundations of Physics*, **20**(7), 823–858.

Stone, M. H. (1936). The theory of representations for a Boolean algebra, *Transactions of the American Mathematical Society*, **40**, 37–111.